

A practical tool to solve indefinite problems: T-coercivity

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- Numerical examples.

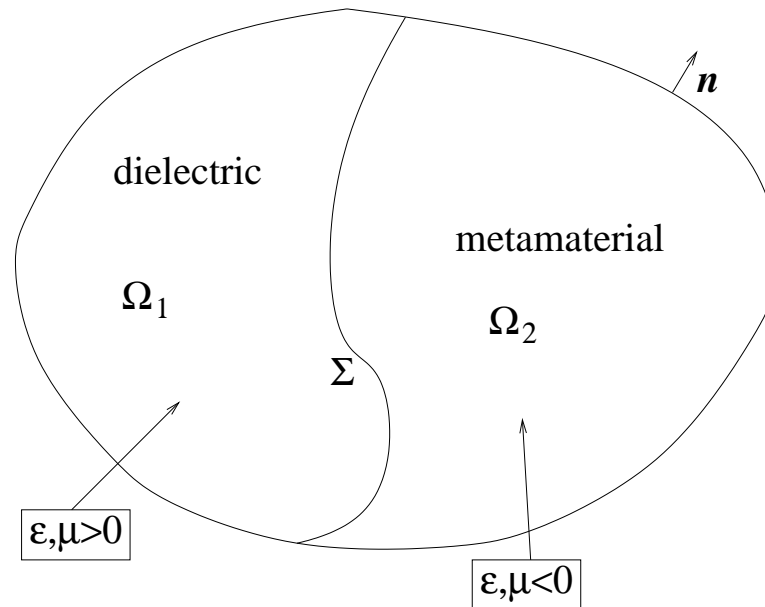
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- Numerical examples.
- Conclusion.

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- *Goal:* Solve numerically a time-harmonic problem in electromagnetism, set in a heterogeneous medium like below.

The domain Ω :

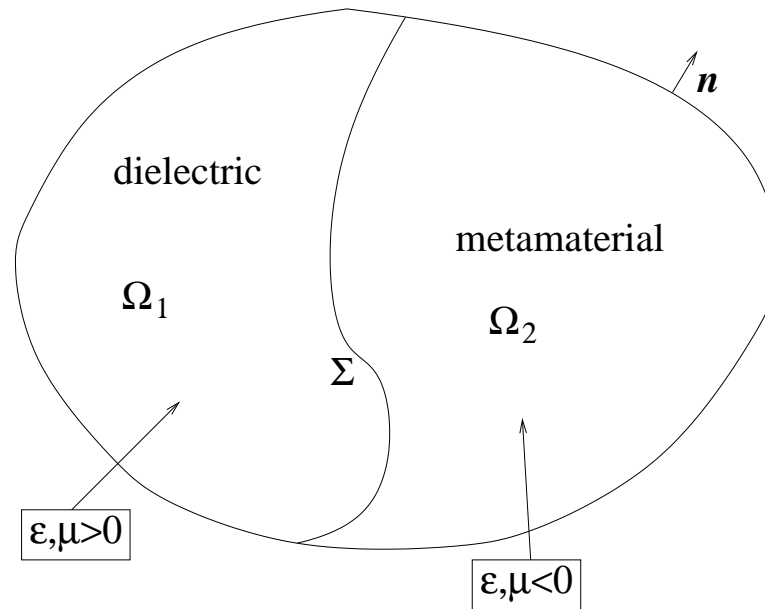


At a given frequency ω , the **(negative) metamaterial** is modelled as a material with **real, strictly negative**, electric permittivity ϵ and magnetic permeability μ .

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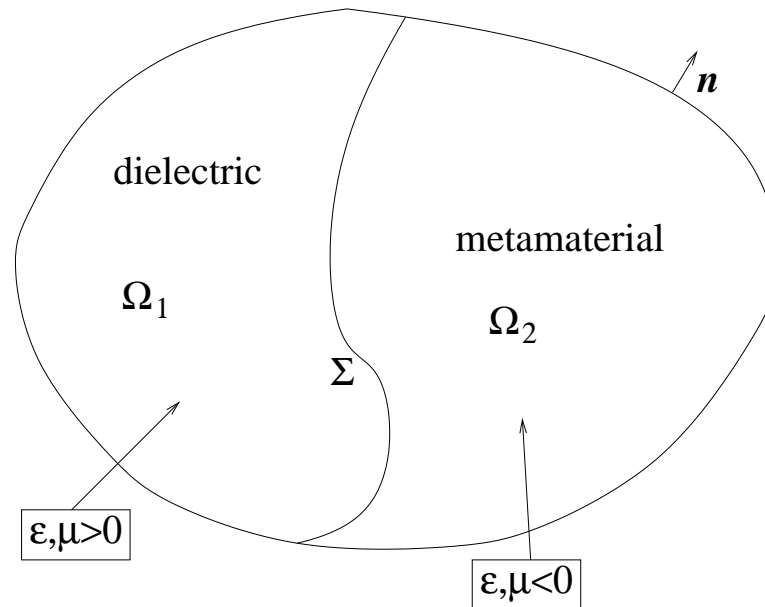


$$\begin{cases} \epsilon := \epsilon_{effective}(\omega) < 0 \\ \mu := \mu_{effective}(\omega) < 0 \end{cases} \text{ in a (negative) metamaterial, in some frequency ranges.}$$

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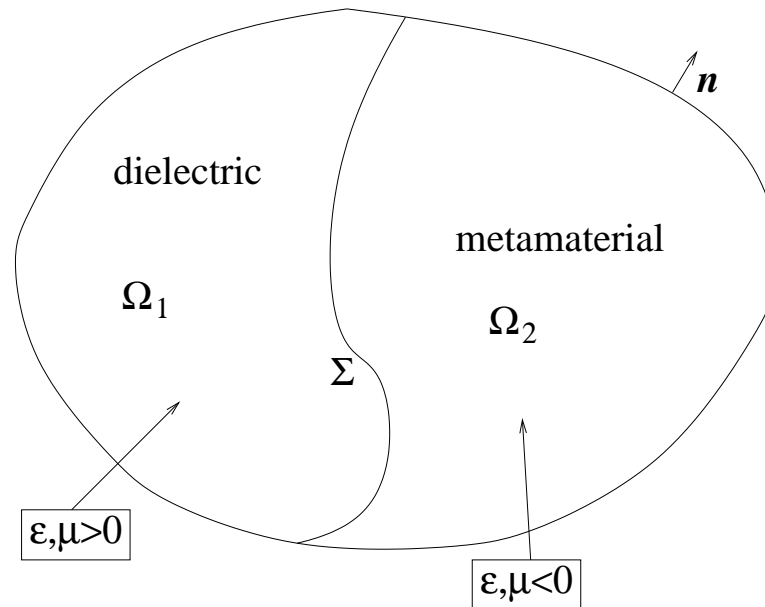
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NB. In general, $\epsilon_{effective}(\omega) = \epsilon' + i\epsilon''$, $(\epsilon', \epsilon'') \in \mathbb{R}^2$, and it can happen that $|\epsilon''| \ll |\epsilon'|$, so we neglect the imaginary part ; similarly for $\mu_{effective}(\omega)$.

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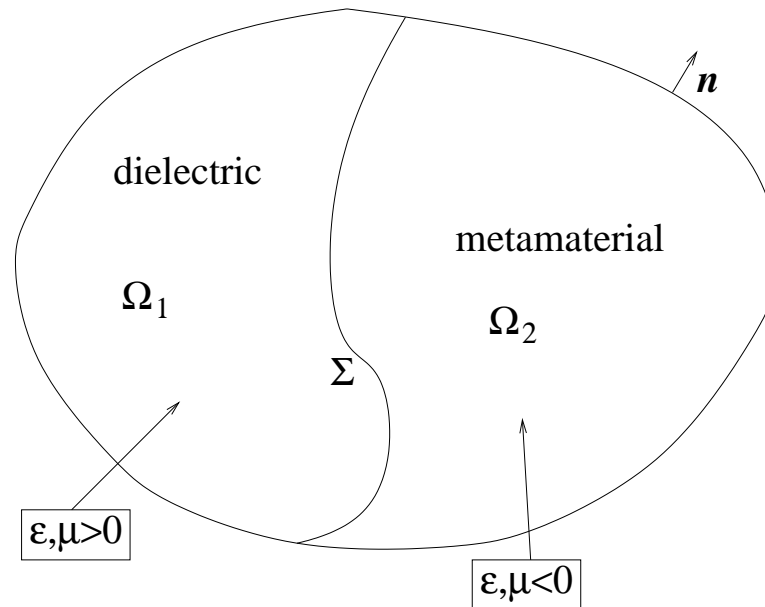


- *Possible practical applications:*
 - perfect lens [Pendry'00], [Maystre-Enoch'04],
 - photonic traps [Genov-Zhang-Zhang'09], etc.

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- *Questions:*
 - Is the problem to be solved *well-posed*?
 - How to *compute* a numerical approximation of the solution?

Maxwell problem (electric field)

Given $\omega > 0$ and source terms $\mathbf{J} \in \mathbf{L}^2(\Omega)$, $\rho \in H^{-1}(\Omega)$ ($\operatorname{div} \mathbf{J} - \omega \rho = 0$).

$$\left\{ \begin{array}{ll} \text{Find } \mathbf{E} \in \mathbf{L}^2(\Omega) \text{ with } \operatorname{curl} \mathbf{E} \in \mathbf{L}^2(\Omega) \text{ s.t.} & \\ \operatorname{curl} (\mu^{-1} \operatorname{curl} \mathbf{E}) - \omega^2 \varepsilon \mathbf{E} = \omega \mathbf{J} & \text{in } \Omega ; \\ \operatorname{div} \varepsilon \mathbf{E} = \rho & \text{in } \Omega ; \\ \mathbf{E} \times \mathbf{n} = 0 & \text{on } \partial\Omega. \end{array} \right.$$

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- Second, set $\mathbf{K} := \omega \mathbf{J} + \omega^2 \varepsilon \operatorname{grad} \varphi$, and solve:

$$(P_E) \quad \left\{ \begin{array}{ll} \text{Find } \mathbf{E}' \in \mathbf{L}^2(\Omega) \text{ with } \operatorname{curl} \mathbf{E}' \in \mathbf{L}^2(\Omega) \text{ s.t.} & \\ \operatorname{curl} (\mu^{-1} \operatorname{curl} \mathbf{E}') - \omega^2 \varepsilon \mathbf{E}' = \mathbf{K} & \text{in } \Omega ; \\ \operatorname{div} \varepsilon \mathbf{E}' = 0 & \text{in } \Omega ; \\ \mathbf{E}' \times \mathbf{n} = 0 & \text{on } \partial\Omega. \end{array} \right.$$

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- The electric field $\mathbf{E} := \mathbf{E}' + \operatorname{grad} \varphi$ solves the Maxwell problem.

Maxwell problem (electric field)-2

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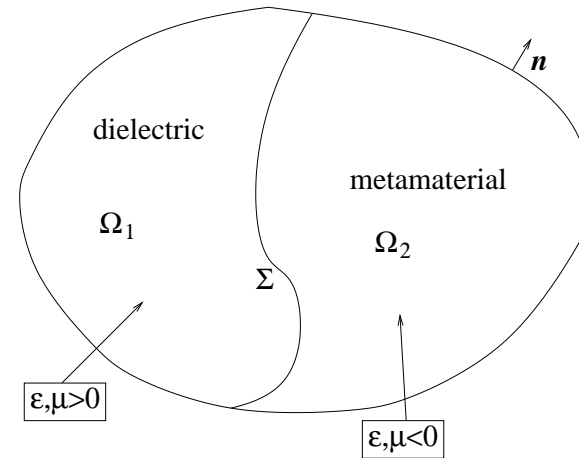
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● Second, in $\mathbf{H}_0(\mathbf{curl}; \Omega) := \{\mathbf{F} \in \mathbf{L}^2(\Omega) \mid \mathbf{curl} \mathbf{F} \in \mathbf{L}^2(\Omega), \mathbf{F} \times \mathbf{n}|_{\partial\Omega} = 0\}$:

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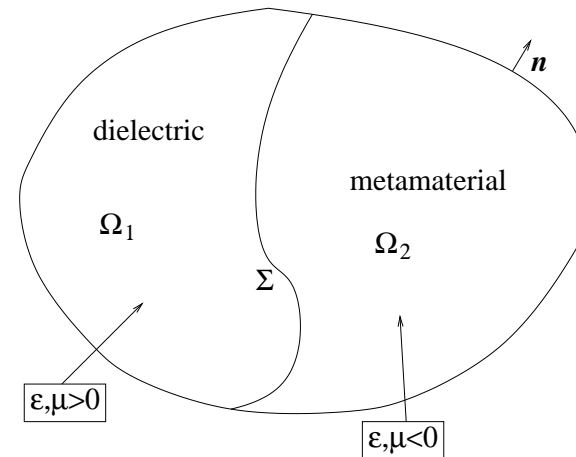
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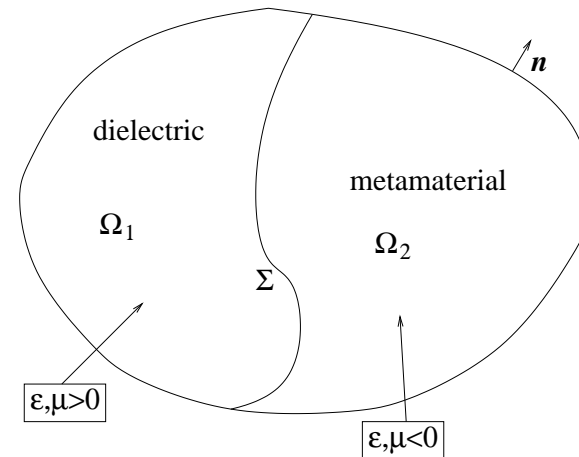


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The form $(\psi, \psi') \mapsto \int_{\Omega} \varepsilon \mathbf{grad} \psi \cdot \mathbf{grad} \psi' d\Omega$ is **indefinite**.

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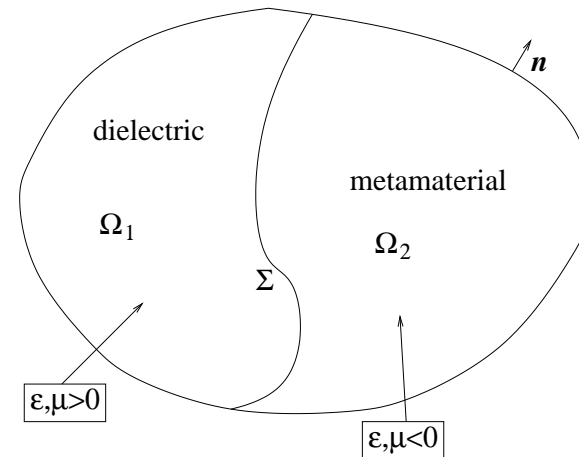


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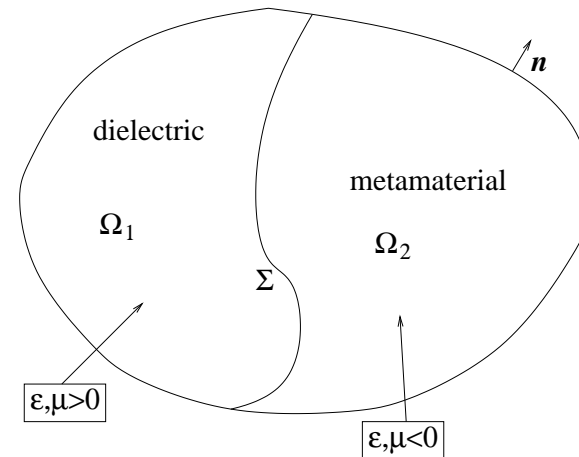
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In addition, what are the properties of the "electric" functional space

$$\mathbf{X}_{\varepsilon}(\Omega) := \{ \mathbf{F} \in \mathbf{H}_0(\mathbf{curl}; \Omega) \mid \operatorname{div} \varepsilon \mathbf{F} \in L^2(\Omega) \} ?$$

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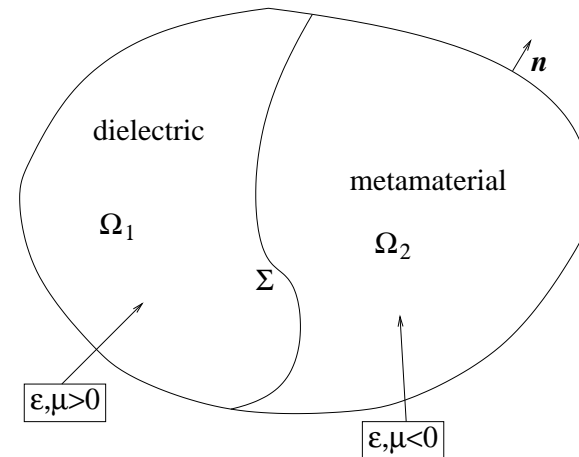
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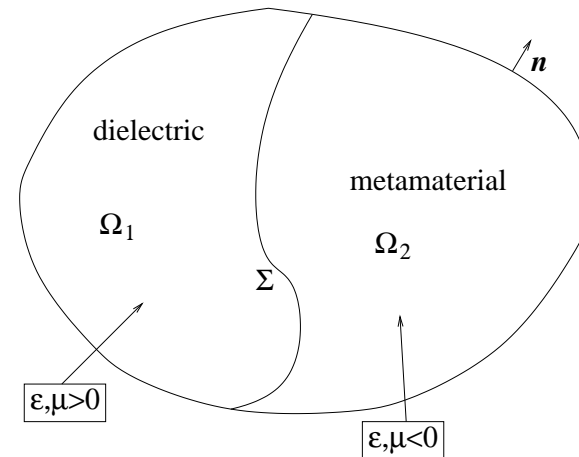
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Then $(\mathbf{F}, \mathbf{F}') \mapsto \omega^2 \int_{\Omega} \varepsilon \mathbf{F} \cdot \overline{\mathbf{F}'} d\Omega$ is treated as a **compact perturbation term**.

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From now on, we focus mainly on the indefiniteness .

Indefinite problems-2

- To fix ideas, we consider Problem (P_φ) , ie. the Variational Formulation

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[Hadamard] The Problem (VF) is *well-posed* if, and only if, for all f , it has one and only one solution u , with continuous dependence:

$$\exists C > 0, \forall f \in V', \|u\|_V \leq C \|f\|_{V'}.$$

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[Lax-Milgram] OK provided that $a(\cdot, \cdot)$ is **coercive!**

Abstract setting-2

● [Banach-Necas-Babuska] Introduce the two conditions

$$(BNB_1) \quad \exists \alpha' > 0, \forall v \in V, \sup_{w \in V \setminus \{0\}} \frac{|a(v, w)|}{\|w\|_V} \geq \alpha' \|v\|_V.$$

$$(BNB_2) \quad \forall w \in V : \{\forall v \in V, a(v, w) = 0\} \implies \{w = 0\}.$$

NB. Condition (BNB_1) is called an *inf-sup condition*, or a *stability condition*.

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- Theorem (Well-posedness) The two assertions below are equivalent:
 - the Problem (VF) is well-posed;
 - the form $a(\cdot, \cdot)$ satisfies conditions (BNB_1) and (BNB_2) .

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- Definition (\mathbb{T} -coercivity) The form $a(\cdot, \cdot)$ is \mathbb{T} -coercive if

$$\exists \mathbb{T} \in \mathcal{L}(V), \text{ bijective}, \exists \underline{\alpha} > 0, \forall v \in V, |a(v, \mathbb{T}v)| \geq \underline{\alpha} \|v\|_V^2.$$

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NB. Condition (BNB_1) is called an *inf-sup condition*, or a *stability condition*.

- Definition (T-coercivity) The form $a(\cdot, \cdot)$ is T-coercive if

$$\exists T \in \mathcal{L}(V), \text{ bijective}, \exists \underline{\alpha} > 0, \forall v \in V, |a(v, Tv)| \geq \underline{\alpha} \|v\|_V^2.$$

- Theorem (Well-posedness) The three assertions below are equivalent:

- the Problem (VF) is well-posed;
- the form $a(\cdot, \cdot)$ satisfies conditions (BNB_1) and (BNB_2) .
- the form $a(\cdot, \cdot)$ is T-coercive.

Abstract setting-3

- Solve the *coercive+compact* Variational Formulation

$$(VF_{c+c}) \left\{ \begin{array}{l} \text{Find } u \in V \text{ s.t.} \\ \forall v \in V, a_0(u, v) + c(u, v) = \langle f, v \rangle, \end{array} \right.$$

with $a_0(\cdot, \cdot)$ and $c(\cdot, \cdot)$ two continuous sesquilinear forms over $V \times V$:

- (c₁) The form $a_0(\cdot, \cdot)$ is **T-coercive**;
- (c₂) The operator $C \in \mathcal{L}(V)$ associated to $c(\cdot, \cdot)$ is **compact**.

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NB. The operator associated to $(a_0 + c)(\cdot, \cdot)$ is **Fredholm of index 0** (and injective).

Practical T-coercivity

● In our case (Problem (VF_φ)):

● Ω, Ω_1 and Ω_2 are domains of \mathbb{R}^d , $d \geq 1$: $\Omega_1 \cap \Omega_2 = \emptyset$, $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$;

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$$V = \{v \mid v|_{\Omega_k} \in V_k, k = 1, 2, \text{Matching}_{\Sigma}(v|_{\Omega_1}, v|_{\Omega_2}) = 0\},$$

$$\text{with } \text{Matching}_{\Sigma}(v_1, v_2) := v_1|_{\Sigma} - v_2|_{\Sigma}.$$

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$$\forall v_1 \in V_1, \sigma_1^- \|\mathbf{grad} v_1\|_{L^2(\Omega_1)}^2 \leq +a_1(v_1, v_1) \leq \sigma_1^+ \|\mathbf{grad} v_1\|_{L^2(\Omega_1)}^2;$$

$$\forall v_2 \in V_2, \sigma_2^- \|\mathbf{grad} v_2\|_{L^2(\Omega_2)}^2 \leq -a_2(v_2, v_2) \leq \sigma_2^+ \|\mathbf{grad} v_2\|_{L^2(\Omega_2)}^2.$$

NB. We assume $0 < \sigma_k^- \leq \sigma_k^+ < \infty, k = 1, 2$.

Practical T-coercivity-2

● First try:

$$\forall v \in H_0^1(\Omega), \quad \mathbb{T}_- v := \begin{cases} v_1 & \text{in } \Omega_1 \\ -v_2 & \text{in } \Omega_2 \end{cases} .$$

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$$\mathbb{T}^2 v = \begin{cases} (\mathbb{T}v)_1 = v_1 & \text{in } \Omega_1 \\ -(\mathbb{T}v)_2 + 2R_1 (\mathbb{T}v)_1 = -(-v_2 + 2R_1 v_1) + 2R_1 v_1 = v_2 & \text{in } \Omega_2 \end{cases}.$$

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Can one achieve T-coercivity?

Practical T-coercivity-3

• Computations:

$$\begin{aligned} |a(v, \mathbf{T}v)| &= |a_1(v_1, v_1) - a_2(v_2, v_2) + 2a_2(v_2, R_1 v_1)| \\ &\geq |a_1(v_1, v_1) - a_2(v_2, v_2)| - 2|a_2(v_2, R_1 v_1)| \\ &\geq \sigma_1^- \|v_1\|_{V_1}^2 - a_2(v_2, v_2) - 2|a_2(v_2, R_1 v_1)| \end{aligned}$$

Practical T-coercivity-3

- Computations: let $\eta > 0$, apply [Young's inequality](#)

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- To obtain $|a(v, \mathbf{T}v)| \geq \underline{\alpha}(\eta) \|v\|_V^2$ for some $\eta > 0$, one needs

$$\frac{\sigma_1^-}{\sigma_2^+} > \|R_1\|^2.$$

Practical T-coercivity-4

● Third try: let $R_2 \in \mathcal{L}(V_2, V_1)$ s.t. for all $v_2 \in V_2$, $\text{Matching}_\Sigma(R_2 v_2, v_2) = 0$.

$$\forall v \in H_0^1(\Omega), \quad \mathbb{T} v := \begin{cases} v_1 - 2R_2 v_2 & \text{in } \Omega_1 \\ -v_2 & \text{in } \Omega_2 \end{cases} .$$

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- Conclusion: to achieve T-coercivity, one needs

$$\frac{\sigma_1^-}{\sigma_2^+} > \left(\inf_{R_1} \|\|R_1\|\| \right)^2 \quad \text{or} \quad \frac{\sigma_2^-}{\sigma_1^+} > \left(\inf_{R_2} \|\|R_2\|\| \right)^2.$$

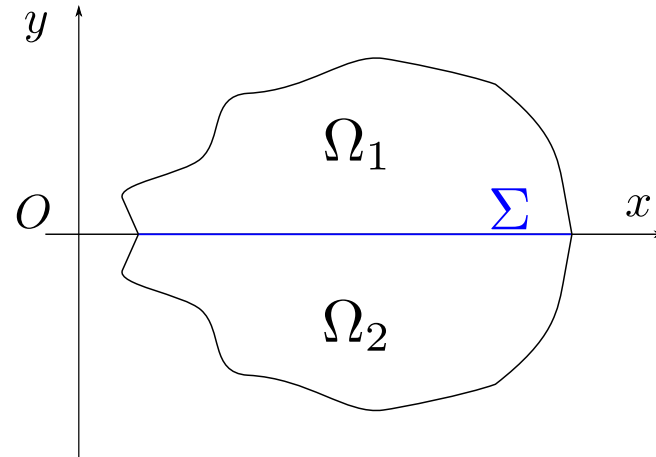
Optimality of T-coercivity

- Study of an elementary setting:
 - piecewise constant coefficient σ ;
in this case, $\sigma_1^- = \sigma_1^+ = \sigma_1$, and $\sigma_2^- = \sigma_2^+ = |\sigma_2|$;
define the *contrast* $\kappa_\sigma = \frac{\sigma_2}{\sigma_1} \in]-\infty, 0[$.

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- First case: $\sigma_1 \neq -\sigma_2$, or $\kappa_\sigma \neq -1$, in a symmetric geometry.

Sample symmetric geometry:



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Let $R_1 \in \mathcal{L}(V_1, V_2)$ s.t. for all $v_1 \in V_1$, $R_1 v_1(x, y) = v_1(x, -y)$, a.e. in Ω_2 .

One finds $\|R_1\| = 1$.

To achieve T-coercivity, one needs $\frac{\sigma_1}{|\sigma_2|} > 1$.

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To achieve T-coercivity, one needs $\frac{|\sigma_2|}{\sigma_1} > 1$.

● Conclusion: Problem (VF_φ) is **well-posed** when $\kappa_\sigma \neq -1$.

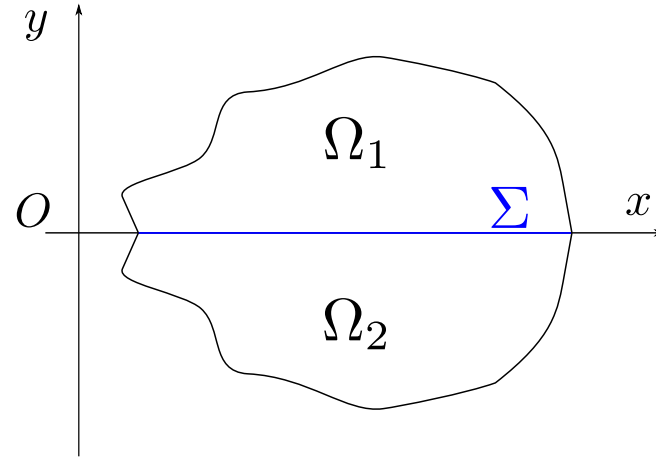
Optimality of T-coercivity-2

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Optimality of T-coercivity-2

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Sample symmetric geometry:



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Let $g \in C_0^\infty(\Sigma)$ and solve for $k = 1, 2$

$$\left\{ \begin{array}{ll} \text{Find } U_k \in H_{0,\Gamma_k}^1(\Omega) \text{ s.t.} & \\ \Delta U_k = 0 & \text{in } \Omega_k ; \\ U_k = g & \text{on } \Sigma. \end{array} \right.$$

Define u by $u|_{\Omega_k} = U_k$, $k = 1, 2$: Matching $_{\Sigma}(u|_{\Omega_1}, u|_{\Omega_2}) = 0$, so $u \in H_0^1(\Omega)$.

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Define u by $u|_{\Omega_k} = U_k$, $k = 1, 2$: Matching $_{\Sigma}(u|_{\Omega_1}, u|_{\Omega_2}) = 0$, so $u \in H_0^1(\Omega)$.

By symmetry, $u(x, -y) = u(x, y)$ a.e. in Ω : so $\sigma_1 \frac{\partial u_1}{\partial y} = \sigma_2 \frac{\partial u_2}{\partial y}$ on Σ .

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- Second case: $\sigma_1 = -\sigma_2$, or $\kappa_\sigma = -1$, in a symmetric geometry.

Let $g \in C_0^\infty(\Sigma)$ and solve for $k = 1, 2$

$$\begin{cases} \text{Find } U_k \in H_{0,\Gamma_k}^1(\Omega) \text{ s.t.} \\ \Delta U_k = 0 & \text{in } \Omega_k ; \\ U_k = g & \text{on } \Sigma. \end{cases}$$

Define u by $u|_{\Omega_k} = U_k$, $k = 1, 2$: Matching $_{\Sigma}(u|_{\Omega_1}, u|_{\Omega_2}) = 0$, so $u \in H_0^1(\Omega)$.

By symmetry, $u(x, -y) = u(x, y)$ a.e. in Ω : so $\sigma_1 \frac{\partial u_1}{\partial y} = \sigma_2 \frac{\partial u_2}{\partial y}$ on Σ .

It follows that $u \in H_0^1(\Omega)$, with $\operatorname{div} \sigma \operatorname{grad} u = 0$ in Ω .

Optimality of T-coercivity-2

- Study of an elementary setting:
 - piecewise constant coefficient σ .
- Second case: $\sigma_1 = -\sigma_2$, or $\kappa_\sigma = -1$, in a symmetric geometry.

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- Conclusion: Problem (VF_φ) is **ill-posed** when $\kappa_\sigma = -1$ (*Critical case.*)

Optimality of T-coercivity-3

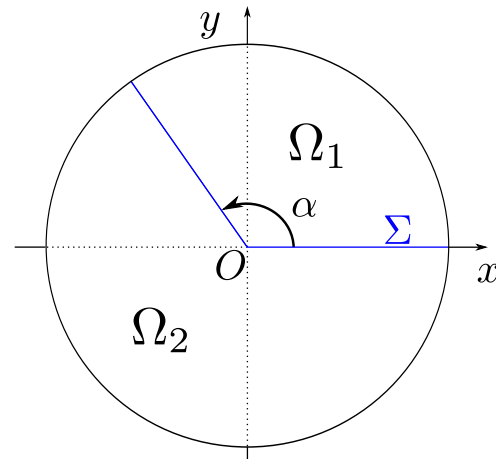
- Simple geometries:
 1. Symmetric geometry

Optimality of T-coercivity-3

● Simple geometries:

1. Symmetric geometry
2. Interface with an interior vertex

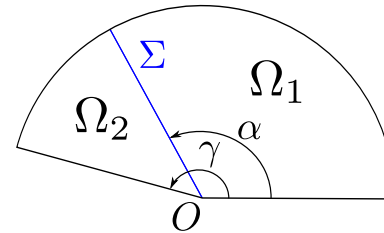
Sample geometry:



Optimality of T-coercivity-3

- Simple geometries:
 1. Symmetric geometry
 2. Interface with an interior vertex
 3. Interface with a boundary vertex

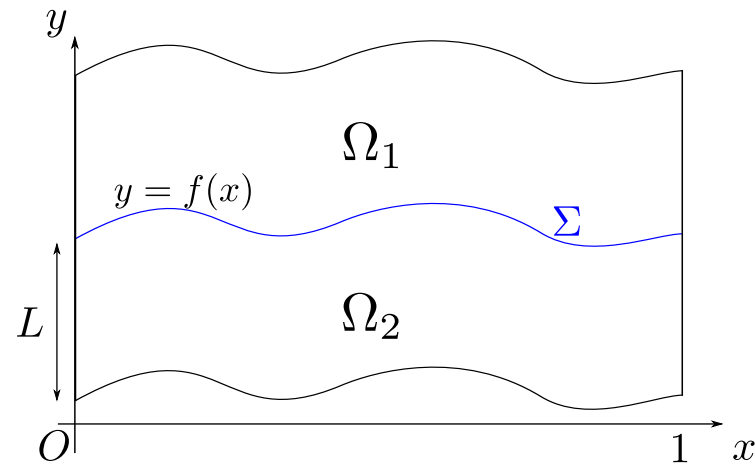
Sample geometry:



Optimality of T-coercivity-3

- Simple geometries:
 1. Symmetric geometry
 2. Interface with an interior vertex
 3. Interface with a boundary vertex
 4. C^1 -class interface

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Optimality of T-coercivity-3

- Simple geometries:
 1. Symmetric geometry
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- Handle general geometries by *localization*.

Optimality of T-coercivity-3

- Simple geometries:

1. Symmetric geometry
2. Interface with an interior vertex
3. Interface with a boundary vertex
4. C^1 -class interface

- Handle general geometries by *localization*.

- Build a partition of unity, and use the T-coercivity results locally.
- *A priori* estimate: there exists an interval I_Σ of $] -\infty, 0[$ s.t. if $\kappa_\sigma \notin I_\Sigma$, then

$$\exists C > 0, \forall v \in H_0^1(\Omega), \|v\|_{H_0^1(\Omega)} \leq C \{ \|\operatorname{div} \sigma \mathbf{grad} v\|_{H^{-1}(\Omega)} + \|v\|_{L^2(\Omega)} \}.$$

- Use **Peetre's Lemma** to conclude.

Optimality of T-coercivity-3

- Simple geometries:
 1. Symmetric geometry
 2. Interface with an interior vertex
 3. Interface with a boundary vertex
 4. C^1 -class interface
- Handle general geometries by *localization*.

If $\kappa_\sigma \notin I_\Sigma$, then Problem (VF_φ) is well-posed in the Fredholm sense.

- In this case, the associated operator is **Fredholm of index 0**.

Optimality of T-coercivity-3

- Simple geometries:
 1. Symmetric geometry
 2. Interface with an interior vertex
 3. Interface with a boundary vertex
 4. \mathcal{C}^1 -class interface
- Handle general geometries by *localization*.

If $\kappa_\sigma \notin I_\Sigma$, then Problem (VF_φ) is well-posed in the Fredholm sense.

- In this case, the associated operator is **Fredholm of index 0**.
- The interval I_Σ always contains -1 .
- If the interface is \mathcal{C}^1 without endpoints, then $I_\Sigma = \{-1\}$.
- Problem (VF_{E_z}) can be solved similarly.

Optimality of T-coercivity-4

- When is the operator
 - Fredholm (of index 0) ;
 - or not Fredholm? (*Critical case.*)

Optimality of T-coercivity-4

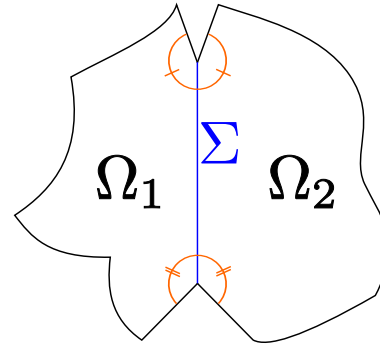
- When is the operator
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When is it Fredholm of index 0?

(use the previous result)

Locally symmetric geometry:

$$\kappa_\sigma \neq -1.$$



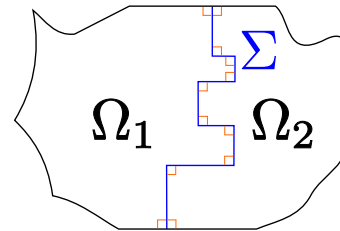
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- When is the operator
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Right angles:
 $\kappa_\sigma \notin [-3, -1/3]$.



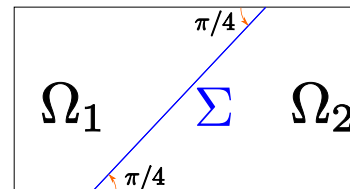
Optimality of T-coercivity-4

- When is the operator
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When is it Fredholm of index 0?

(use the previous result)

Boundary vertices with
angles $\pi/4$ and $3\pi/4$:
 $\kappa_\sigma \notin [-3, -1/3]$.



Optimality of T-coercivity-4

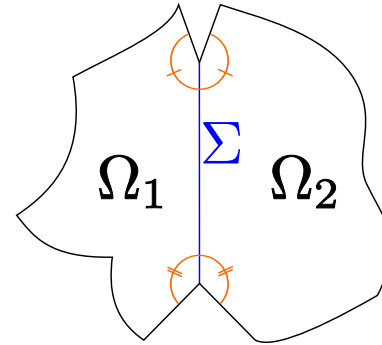
- When is the operator
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When is it not Fredholm?

(direct computations: line singularity)

Locally symmetric geometry:

$$\kappa_\sigma = -1.$$



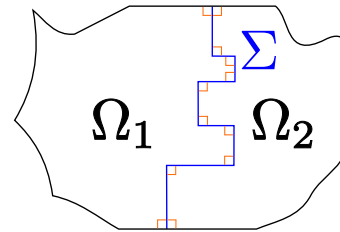
Optimality of T-coercivity-4

- When is the operator
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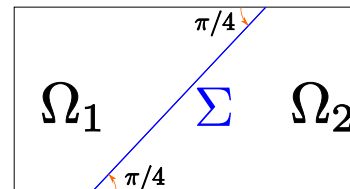
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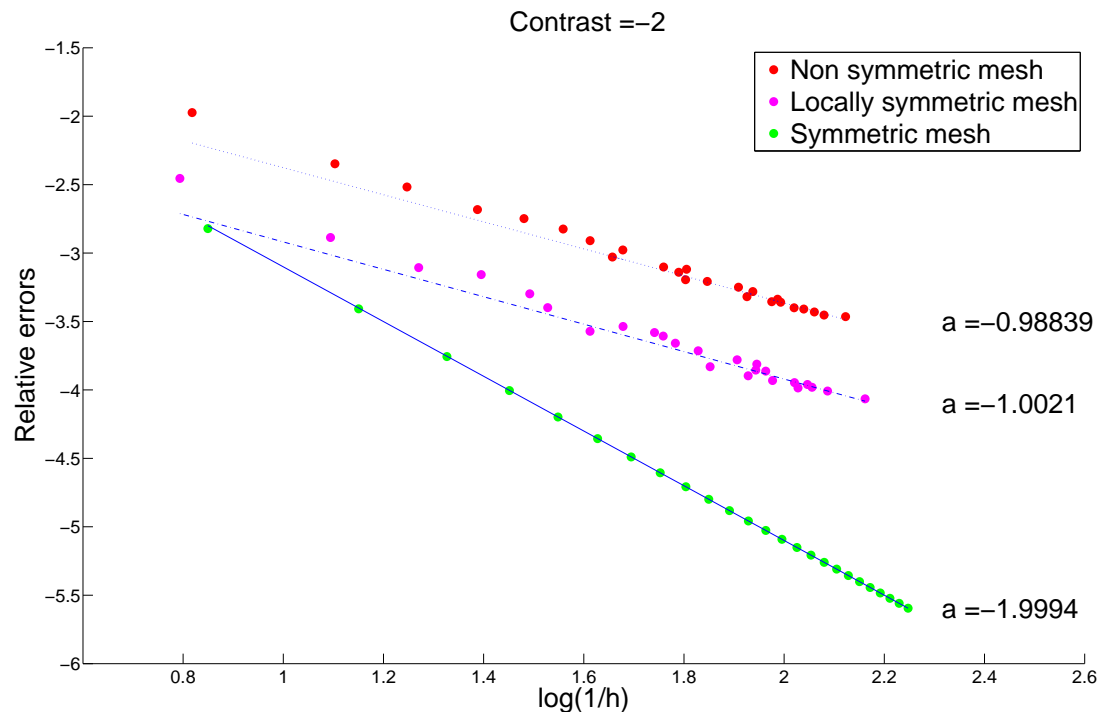


Numerical experiments

- In a symmetric domain, made up of two adjacent squares.
- An *exact* piecewise smooth solution of Problem (VF_φ) is available.
- Two contrasts: $\kappa_\sigma \in \{-2, -1.001\}$.
- Discretization using P_1 Lagrange FE.
- We study below the influence of the meshes.

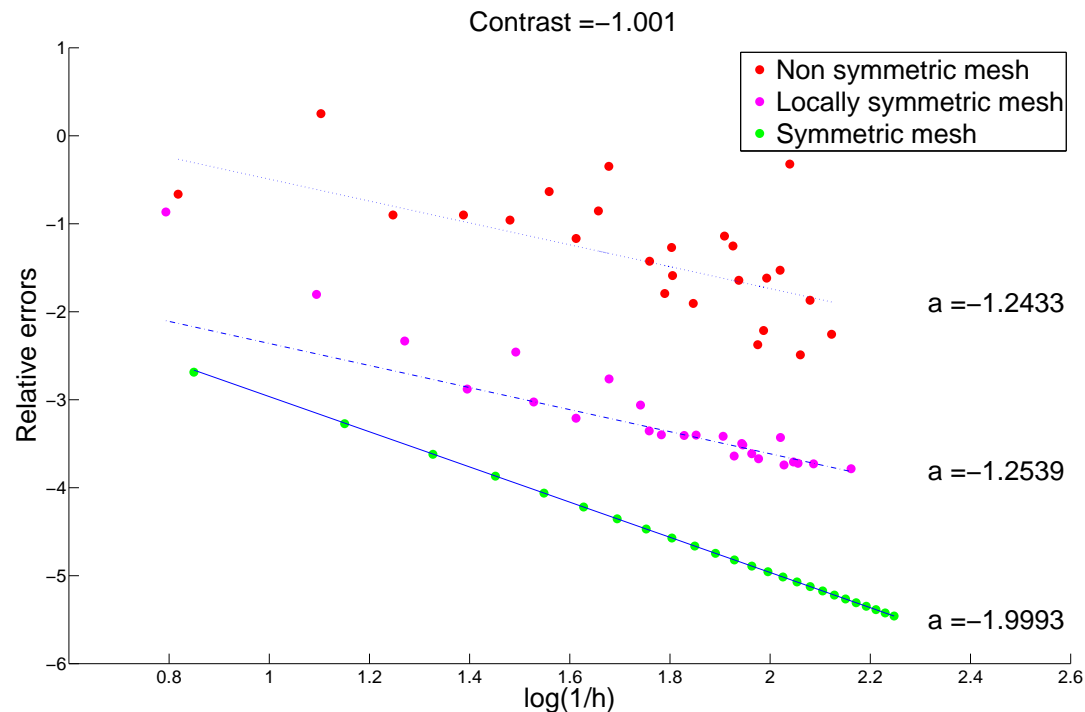
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- Contrast $\kappa_\sigma = -2$:



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- Contrast $\kappa_\sigma = -1.001$:



Numerical experiments-2

- In the unit cube, split in two halves (with $\Sigma := \{\frac{1}{2}\} \times]0, 1[\times]0, 1[$).
- Piecewise constant ε, μ .
- An *exact* piecewise smooth solution of Maxwell's equations is available.
- Discretization of the *augmented* formulation [Jr'05]

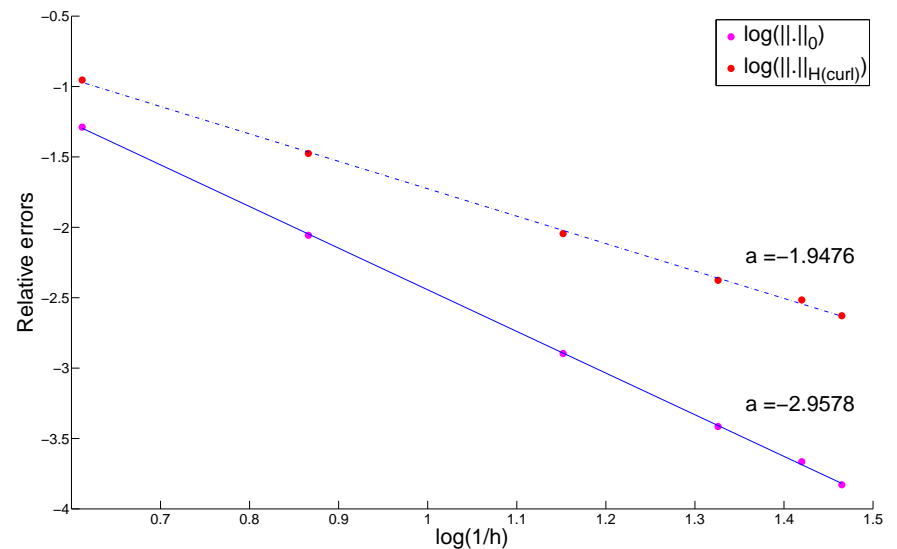
(set in $\mathbf{X}_\varepsilon(\Omega) = \{\mathbf{F} \in \mathbf{H}_0(\mathbf{curl}; \Omega) \mid \operatorname{div} \varepsilon \mathbf{F} \in L^2(\Omega)\}$.)

$$\left\{ \begin{array}{l} \text{Find } \mathbf{E}' \in \mathbf{X}_\varepsilon(\Omega) \text{ s.t.} \\ \forall \mathbf{F} \in \mathbf{X}_\varepsilon(\Omega), \quad \int_{\Omega} \mu^{-1} (\mathbf{curl} \mathbf{E}' \cdot \overline{\mathbf{curl} \mathbf{F}} + \varepsilon^{-2} \operatorname{div} \varepsilon \mathbf{E}' \overline{\operatorname{div} \varepsilon \mathbf{F}}) d\Omega \\ \qquad \qquad \qquad -\omega^2 \int_{\Omega} \varepsilon \mathbf{E}' \cdot \overline{\mathbf{F}} d\Omega = \int_{\Omega} \mathbf{K} \cdot \overline{\mathbf{F}} d\Omega . \end{array} \right.$$

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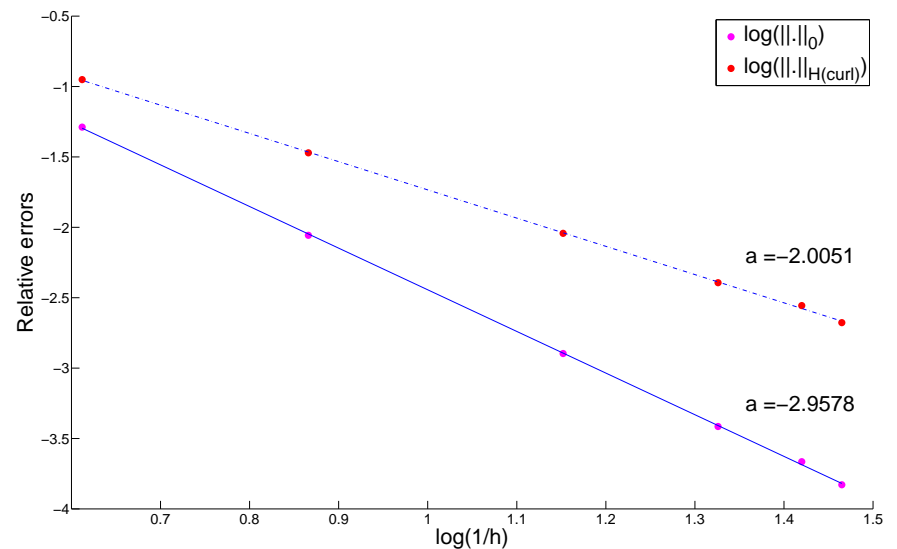
"Usual" case: $\omega = 4$,
 $(\varepsilon_1, \mu_1) = (+1, +1)$,
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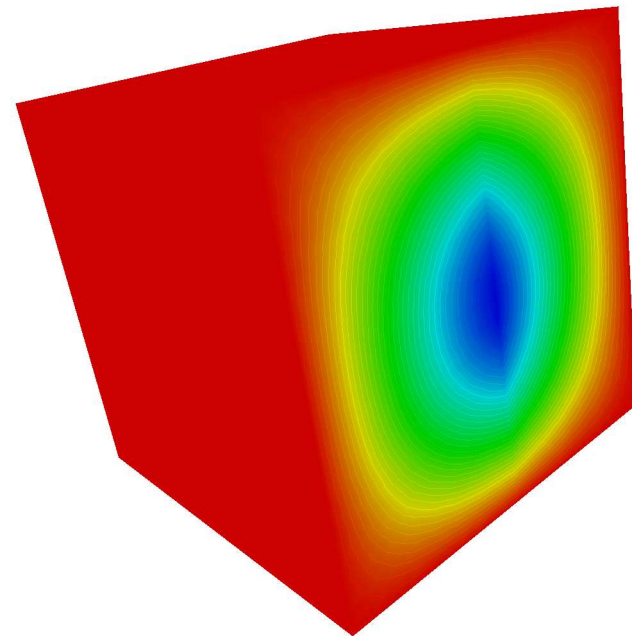
"Unusual" case: $\omega = 4$,
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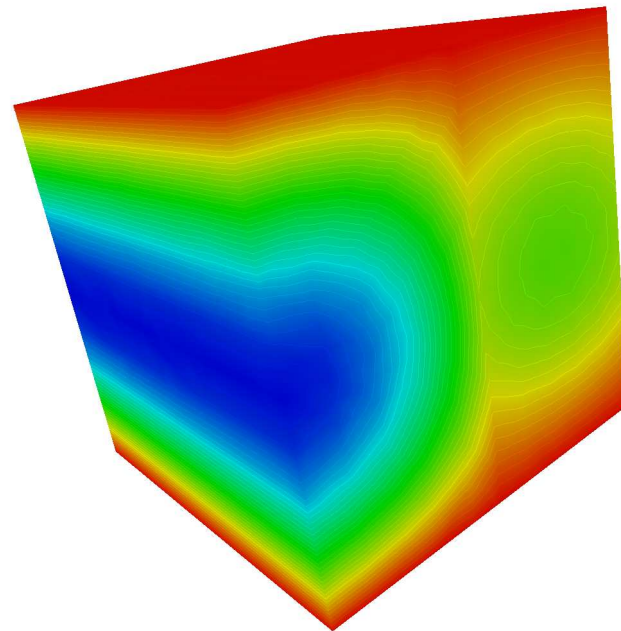
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computed electric field (\mathbf{E}_y^h).



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computed magnetic field (\mathbf{H}_z^h).



Conclusions/Perspectives

- For the scalar problems:
 - numerical analysis when Γ -coercivity applies (cf. [BonnetBenDhia-Jr-Zwölf'10], [Nicaise-Venel'11], DG-approach [Chung-Jr'11], etc.) ;
 - theoretical study of the *critical* cases (with X. Claeys (ISAE)) ;
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 - work out the theory of T-coercivity (side results: compact imbedding(s), etc.) ;
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- For the Maxwell problem(s):
 - work out the theory of T-coercivity (side results: compact imbedding(s), etc.) ;
 - numerical analysis when T-coercivity applies.

- In the *critical* cases: are models derived from physics still relevant?
 - re-visit models (homogenization, multi-scale numerics, etc.).
(METAMATH Project, submitted to ANR ; coordinator S. Fliss (POEMS)).