Nonhomogeneous nilpotent approximations for systems with singularities

Marilena Vendittelli, Giuseppe Oriolo, Frédéric Jean, Jean-Paul Laumond

Abstract

Nilpotent approximations are a useful tool for analyzing and controlling systems whose tangent linearization does not preserve controllability, such as nonholonomic mechanisms. However, the conventional *homogeneous* nilpotent approximations exhibit a drawback: in the neighborhood of singular points (where the system growth vector is not constant) the vector fields of the approximate dynamics do not vary continuously with the approximation point. The geometric counterpart of this situation is that the sub-Riemannian distance estimate provided by the classical Ball-Box Theorem is not uniform at singular points. With reference to a specific family of driftless systems, which includes an interesting example of nonholonomic mechanism called the general 2-trailer system, we show how to build a nonhomogeneous nilpotent approximation whose vector fields vary continuously around singular points. It is also proven that the privileged coordinates associated to such an approximation provide a uniform estimate of the distance.

Keywords

Nilpotent approximations, nonholonomic systems, approximate steering, singularities, sub-Riemannian distance.

I. Introduction

Studying local properties of nonlinear systems through some approximation of the original dynamics is often the only viable approach to the solution of difficult synthesis problems. In view of the extensive theory available for linear systems, tangent linearization is the most common procedure. However, this kind of approximation does not always preserve the structural properties of the system; a well-known example of this situation are nonholonomic mechanisms, whose tangent linearization is never controllable. To deal with this case, it is convenient to resort to a *nilpotent approximation* (NA), a higher-order approximation with an increased degree of adherence to the original dynamics. For example, it has been shown in [11] that NAs may be used to evaluate the complexity of nonholonomic motion planning problems. From the viewpoint of synthesis, approximate steering laws computed based on NAs can be instrumental (in conjunction with iterative steering techniques) in the solution of steering and stabilization problems for certain non-nilpotentizable systems [6], [22].

Various researchers have developed techniques for computing homogeneous NAs (e.g., [1], [2], [7], [17]). Essentially, these techniques require first to express the original dynamics in a privileged coordinate system centered at the approximation point, and defined on the basis of the control Lie algebra associated to the input vector fields. Then, the new vector fields are expanded in Taylor series; by truncating the expansion at a proper order, one obtains a nilpotent system which is polynomial and triangular by construction. In the presence of singular points (i.e., points where the growth vector of the system changes), homogeneous NAs exhibit a *discontinuous* dependence

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on the expansion point. In fact, both the privileged coordinate system and the truncation order change at singular points, and thus the vector fields of the approximation vary discontinuously.

Continuity, however, is a critical issue when NAs are used to design approximate steering laws for systems with singularities. An essential step of this study is in fact the estimation of the sub-Riemannian distance d for the approximation. The classical Ball-Box Theorem [2] provides a local estimate of d which depends on the value of privileged coordinates and growth vector at the point. When approaching a singular point through a sequence of regular points, however, the region of validity for such estimate tends to zero [12]. As a result, no uniform estimate of the sub-Riemannian distance is available.

In this note, we show that the above difficulty can be solved by giving up the homogeneity property. With reference to a specific family of nonlinear systems, namely 5-dimensional 2-input driftless systems with generic singularities, we show how to build *nonhomogeneous* NAs that vary continuously with the expansion point over a finite number of subsets that cover the singular locus. By doing so, we can associate to each point of the state space a continuous approximation procedure. As a byproduct, a uniform estimate of the sub-Riemannian distance is also obtained.

We point out that homogeneity may be of crucial importance for preserving controllability and stability: in this sense, a nonhomogeneous nilpotent approximation may even have opposite properties with respect to the original system [9]. For this reason, homogeneity has been assumed in most classical applications of NAs, such as the derivation of sufficient controllability (STLC) conditions for systems with drift [4], [21] and the study of stabilizability for non-smoothly stabilizable systems [10], [18]. However, the arguments to be presented in this note will show that the lack of homogeneity is no complication for sub-Riemannian distance estimation and — consequently — path planning purposes.

II. Background material

The objective of this section is to introduce some basic tools used in sub-Riemannian geometry (e.g., see [1], [7], [17], [20]). In our presentation, we follow [2]. Although the general framework is that of differentiable manifolds, the local nature of our study allows the restriction to $I\!R^n$.

Consider a driftless control system

$$\dot{x} = \sum_{i=1}^{m} g_i(x)\, u_i, \qquad x \in I\!R^n, \tag{1}$$

where $g_1, \ldots, g_m$ are C^∞ vector fields on $I\!R^n$ and the input vector $u(t) = (u_1(t), \ldots, u_m(t))$ takes values on $I\!R^m$. Given $x_0 \in I\!R^n$, let η be a trajectory of (1) originating from x_0 under an input function $u(t)$, $t \in [0, T]$. We define its *length* as

$$\text{length}(\eta) = \int_0^T \sqrt{u_1^2(t) + \ldots + u_m^2(t)}\, dt.$$

A point x such that $x = \eta(t)$, for some $t \in [0, T]$, is said to be *accessible* from x_0.

System (1) induces a *sub-Riemannian distance* d (also known as *Carnot-Carathéodory distance*) on $I\!R^n$, defined as

$$d(x_1, x_2) = \inf_{\eta} \text{length}(\eta), \tag{2}$$

where the infimum is taken over all trajectories η joining x_1 to x_2. Note that $d(x_1, x_2) \leq \infty$ if and only if x_1 and x_2 are accessible from each other.

Chow's Theorem states that any two points in $I\!R^n$ are accessible from each other if the elements of the Lie Algebra $\mathcal{L}_g$ generated by the g_i's span the tangent space $T_{x_0}\mathcal{M}$ at each point x_0. Chow's condition is also known as the Lie Algebra Rank Condition (LARC). As system (1) is driftless, the LARC implies controllability in any usual sense [21]. Throughout this note, we assume that the LARC is satisfied.

Take $x_0 \in I\!R^n$ and let $L^s(x_0)$ be the vector space generated by the values at x_0 of the brackets of $g_1, \ldots, g_m$ of length $\leq s$, $s = 1, 2, \ldots$ (input vector fields are brackets of length 1). The LARC guarantees that there exists a smallest integer $r = r(x_0)$ such that $\dim L^r(x_0) = n$. This integer is called the *degree of nonholonomy* at x_0.

Let $n_s(x) = \dim L^s(x)$, $s = 1, \ldots, r$. A point x_0 is said to be *regular* if the *growth vector* $(n_1(x), \ldots, n_r(x))$ remains constant in a neighborhood of x_0; otherwise x_0 is *singular*. In particular, points at which the degree of nonholonomy changes are certainly singular. Regular points form an open and dense set in $I\!R^n$.

A. Nilpotent Approximations and Privileged Coordinates

Consider a smooth real-valued function f. Call *first-order nonholonomic partial derivatives* of f the Lie derivatives $g_i f$ of f along g_i, $i = 1, \ldots, m$. Call $g_i(g_j f)$, $i, j = 1, \ldots, m$, the *second-order nonholonomic partial derivatives* of f, and so on.

Definition 1: A function f is said to be *of order* $\geq s$ at a point x_0 if all its nonholonomic partial derivatives of order $\leq s-1$ vanish at x_0. If f is of order $\geq s$ and not of order $\geq s+1$ at x_0, it is said to be *of order* s at x_0.

An equivalent characterization of this notion is the following: if f is of order s at x_0, then $f(x) = O(d^s(x_0, x))$, and viceversa.

Definition 2: A vector field h is said to be *of order* $\geq q$ at x_0 if, for every s and every function f of order s at x_0, the function hf has order $\geq q + s$ at x_0. If h is of order $\geq q$ but not $\geq q+1$, it is said to be *of order* q at x_0.

Using these definitions, it is easy to show that input vector fields g_i, $i = 1, \ldots, m$, have order ≥ -1, brackets $[g_i, g_j]$, $i, j = 1, \ldots, m$, have order ≥ -2, and so on.

Definition 3: A system

$$\dot{x} = \sum_{i=1}^{m} \hat{g}_i(x)\, u_i,$$

defined on a neighborhood of x_0, is a *nilpotent approximation* (NA) of system (1) at x_0 if

a) the vector fields $g_i - \hat{g}_i$ are of order ≥ 0 at x_0;

b) its the Lie algebra is *nilpotent* of step $s > r(x_0)$, i.e., all Lie brackets of length greater than s vanish.

This definition is equivalent to that given in [2], [8]. In particular, the property *a)* implies the preservation of growth vector and LARC.

Algorithms for computing nilpotent approximations are based on the existence at each point of a set of locally defined privileged coordinates.

Definition 4: Let the integer w_j, $j = 1, \ldots, n$, be defined by setting $w_j = s$ if $n_{s-1} < j \leq n_s$, with $n_s = n_s(x_0)$ and $n_0 = 0$. The local coordinates $z_1, \ldots, z_n$ centered at x_0 form a system of *privileged coordinates* if the order of z_j at x_0 equals w_j (called the *weight* of coordinate z_j), for $j = 1, \ldots, n$.

The order of functions and vector fields expressed in privileged coordinates can be computed in an algebraic way:

• The order of the monomial $z_1^{\alpha_1} \dots z_n^{\alpha_n}$ is equal to its weighted degree $w(\alpha) = w_1\alpha_1 + \dots + w_n\alpha_n$.

• The order of a function $f(z)$ at $z = 0$ (the image of x_0) is the least weighted degree of the monomials actually appearing in the Taylor expansion of f at 0.

• The order of a vector field $h(z) = \sum_{j=1}^n h_j(z)\partial_{z_j}$ at $z = 0$ is the least weighted degree of the monomials actually appearing in the Taylor expansion of h at 0:

$$h(z) \sim \sum_{\alpha,j} a_{\alpha,j} z_1^{\alpha_1} \dots z_n^{\alpha_n} \partial_{z_j},$$

considering the term $a_\alpha z_1^{\alpha_1} \dots z_n^{\alpha_n} \partial_{z_j}$ as a monomial and assigning to ∂_{z_j} the weight $-w_j$.

For our developments it is convenient to define the notion of approximation procedure.

Definition 5: An *approximation procedure* of system (1) on a given open domain $\mathcal{V} \subset I\!R^n$ is a function AP which associates to each point $x_0 \in \mathcal{V}$ a smooth mapping $z : I\!R^n \to I\!R^n$ and a driftless control system Ψ on $I\!R^n$, given by the vector fields $\hat{g}_1, \dots, \hat{g}_m$, such that:

• $z = (z_1, \dots, z_n)$ restricted to a neighborhood Ω of x_0 forms a system of privileged coordinates at x_0;

• the pull-backs $z^*\hat{g}_i$ of the vector fields $\hat{g}_i$ by z define on Ω a nilpotent approximation of (1) at x_0.

In other words, Ψ is a NA at 0 of (1) expressed in the z coordinates. One example of such procedure is recalled below.

B. Homogeneous Approximation Procedure

Consider system (1) and an approximation point $x_0 \in I\!R^n$. An algorithm for computing a set of privileged coordinates and a nilpotent approximation at x_0 is the following [2].

1. Compute the growth vector and the weights $w_1, \dots, w_n$ as described in the previous section.
2. Choose vector fields $\gamma_1, \dots, \gamma_n$ such that their values at x_0 form a basis of $L^r(x_0) = T_{x_0} I\!R^n$ and

$$\gamma_{n_{s-1}+1}(x), \dots, \gamma_{n_s}(x) \in L^s(x), \quad s = 1, \dots, r,$$

for any x in a neighborhood of x_0.

3. From the original coordinates $x = (x_1, \dots, x_n)$, compute local coordinates $y = (y_1, \dots, y_n)$ as

$$y = \Gamma^{-1}(x - x_0),$$

where Γ is the $n \times n$ matrix whose elements Γ_{ij} are defined by

$$\gamma_j(x_0) = \sum_{i=1}^n \Gamma_{ij} \partial_{x_i}|_{x_0}.$$

4. Build a system of privileged coordinates $z = (z_1, \dots, z_n)$ around x_0 via the recursive formula[1]

$$z_j = y_j + \sum_{k=2}^{w_j-1} h_k(y_1, \dots, y_{j-1}), \qquad j = 1, \dots, n \tag{3}$$

[1]Note that the formula for building privileged coordinates given in [3] is not complete.

where

$$h_k(y_1,\dots,y_{j-1}) = -\sum_{\substack{|\alpha|=k\\ w(\alpha)<w_j}} \gamma_1^{\alpha_1}\dots\gamma_{j-1}^{\alpha_{j-1}}(y_j+\sum_{q=2}^{k-1}h_q)(x_0)\prod_{i=1}^{j-1}\frac{y_i^{\alpha_i}}{\alpha_i!},$$

with $|\alpha| = \sum_{i=1}^n \alpha_i$.

5. Express the dynamics of the original system in privileged coordinates:

$$\dot z = \sum_{i=1}^m g_i(z)u_i.$$

6. Expand the vector fields $g_i(z)$ in Taylor series at 0, express them in terms of vector fields that are homogeneous with respect to the weighted degree

$$g_i(z) = g_i^{(-1)}(z) + g_i^{(0)}(z) + g_i^{(1)}(z) + \dots\ ,$$

and let $\hat g_i(z) = g_i^{(-1)}(z)$.

7. Define the approximate system as

$$\dot z_j = \sum_{i=1}^m \hat g_{ij}(z_1,\dots,z_{j-1})u_i, \quad j=1,\dots,n, \tag{4}$$

where the $\hat g_{ij}$'s (the components of $\hat g_i(z)$) are homogeneous polynomials of weighted degree $w_j - 1$.

System (4) is a NA (triangular by construction) of the original dynamics (1) in the z coordinates, hereafter referred to as a *homogeneous* NA.

Note that, strictly speaking, the above algorithm does not represent an approximation procedure because Step 2 contains a choice. Assume, however, that the state space can be covered by a finite set $\{\mathcal{V}_i, i=1,\dots,N\}$ of open domains, such that in each domain $\mathcal{V}_i$ a unique way can be specified for choosing the vector fields γ_j, $j=1,\dots,n$, on $\mathcal{V}_i$. By doing so, one obtains an approximation procedure AP_i for each domain. An example of this construction will be given in Sect. IV-A.

C. Distance Estimation

Privileged coordinates provide an estimate of the sub-Riemannian distance d, according to the following result.

Ball-Box Theorem: Consider system (1) and a set of privileged coordinates $z=(z_1,\dots,z_n)$ at x_0. There exist positive constants c_0, C_0 and ϵ_0 such that, for all x with $d(x_0,x) < \epsilon_0$,

$$c_0 f(z) \le d(x_0,x) \le C_0 f(z), \tag{5}$$

where $f(z) = |z_1|^{1/w_1} + \dots + |z_n|^{1/w_n}$.

III. Objective

Assume we wish to control system (1) around a point $\bar x$ by means of nilpotent approximations computed in the vicinity of $\bar x$ (for example, using this tool in conjunction with the iterative steering technique of [15], as proposed in [22]). To this end, we use an approximation procedure AP defined on an open domain $\mathcal{V}$ including $\bar x$ to compute a NA Ψ and the associated privileged coordinates z at $x_0 \in \mathcal{V}$.

To guarantee that the structure of the NA does not change in $\mathcal{V}$ — a fact which would hinder its use for control synthesis — it is essential that AP is a *continuous*[2] function in $\mathcal{V}$. That is, both the privileged coordinates and the nilpotent approximation must vary continuously with respect to the approximation point $x_0 \in \mathcal{V}$. If $\bar{x}$ is a regular point, the homogeneous approximation procedure clearly satisfies this requirement; however, when $\bar{x}$ is singular, the growth vector and the associated privileged coordinates weights change around the point, implying that the homogeneous approximation procedure is discontinuous at $\bar{x}$.

A similar difficulty arises when considering distance estimation based on privileged coordinates. Around a regular point $\bar{x}$, coordinates z and constants c_0, C_0 and ϵ_0 depend continuously on the approximation point x_0. This is not true at a singular point. In particular, if $\{x_i\}$ is a sequence of regular points converging to a singular point x_∞, then ϵ_i tends to 0 although ϵ_∞ is nonzero. Hence, if $\bar{x}$ is singular, the estimate (5) does not hold *uniformly* in $\mathcal{V}$; that is, there is no $\epsilon > 0$ such that the estimate holds for any x_0 and x in $\mathcal{V}$ that satisfy $d(x_0, x) < \epsilon$.

The objective of this note may now be stated. With reference to a particular family of 5-dimensional driftless systems with singularities, it will be shown that, in a neighborhood of each point $\bar{x} \in I\!R^5$ it is possible to define an approximation procedure which is continuous at $\bar{x}$. In particular, we shall prove that there exists a finite set of continuous approximation procedures with open domains of definition covering $I\!R^5$. As a consequence, we shall also obtain a modified version of the Ball-Box Theorem yielding an estimate of the sub-Riemannian distance which is uniform with respect to the approximation point x_0. Apart from its intrinsic significance, we expect that this latter development will be essential in deriving (along the lines of [2, Prop. 7.29]) a uniform estimate of the steering error arising from the use of NAs, and ultimately in proving the effectiveness of iterative steering techniques for nonholonomic systems with singularities.

IV. A Family of Systems with Singularities

Consider the 5-dimensional, 2-input, driftless system

$$\dot{x} = g_1(x)u_1 + g_2(x)u_2, \qquad x \in I\!R^5, \tag{6}$$

with g_1, $g_2 \in C^\infty$, and suppose that the system is controllable. Assume further that the growth vector is (2,3,5) at regular points and (2,3,4,5) at singular points; that is, the degree of nonholonomy is 3 at regular points and 4 at singular points. A generic (for the C^∞ Whitney topology) pair of vector fields in $I\!R^5$ satisfies this hypothesis, except possibly on a set of codimension ≥ 4. The kinematic model of the mobile robot system to be presented later falls exactly in the above class.

Applying the necessary and sufficient conditions of [16], one verifies that system (6) cannot be transformed in *chained form* at regular points, due to the 'double jump' in the growth vector[3]. In view of the absence of drift, this also indicates that the system is not *flat*; alternatively, one may directly check that the necessary and sufficient conditions [19] for flatness are violated. If the particular instance of system (6) under consideration is exactly nilpotentizable, one may use the algorithm of [14] to achieve exact steering between arbitrary points; otherwise, no exact steering methods are available. Therefore, it is in general of interest to define an approximation procedure of system (6), either to devise approximate steering techniques or for distance estimation.

[2] The function AP takes values in the product of the set of smooth mappings from $I\!R^n$ to itself and of the set of m-tuples of smooth vector fields on $I\!R^n$, which can be equipped with the product topology induced by the C^0 topology on $C^\infty(I\!R^n)$. The continuity of AP is relative to this topology.

[3] System (6) represents an example of E. Cartan's famous problem of three constraints and five variables [5].

A. Homogeneous Approximation Procedure

First, we recall some algebraic machinery introduced in [21]. Denote by $L(X_1, X_2)$ the free Lie algebra in the *indeterminates* $\{X_1, X_2\}$. The following brackets are the first 8 elements of a P. Hall basis of $L(X_1, X_2)$:

$$\begin{aligned}
&X_1, \;\; X_2,\\
&X_3 = [X_1, X_2],\\
&X_4 = [X_1, [X_1, X_2]], \;\; X_5 = [X_2, [X_1, X_2]],\\
&X_6 = [X_1, [X_1, [X_1, X_2]]], \;\; X_7 = [X_2, [X_1, [X_1, X_2]]], \;\; X_8 = [X_2, [X_2, [X_1, X_2]]].
\end{aligned}$$

Consider system (6), and let E_g be the *evaluation map* which assigns to each $P \in L(X_1, X_2)$ the vector field obtained by plugging in g_i for the corresponding indeterminate X_i $(i = 1, 2)$. The vector fields $g_3, \ldots, g_8$ are given by $g_j = E_g(X_j)$, $j = 3, \ldots, 8$.

Denote by $\mathcal{V}_r$ the open set of regular points, where the growth vector is (2,3,5). In each point of $\mathcal{V}_r$, a basis of the $T_{x_0}\mathbb{R}^5$ is given by the value of

$$B_r = \{g_1, \ldots, g_4, g_5\}.$$

At a singular point, where the growth vector is (2,3,4,5), we need one bracket of length 3 and one of length 4 to span the tangent space. Candidate bases are then given by the value at the point of the following sets of vector fields:

$$B_{ij} = \{g_1, g_2, g_3, g_i, g_j\}, \quad i = 4, 5, \quad j = 6, 7, 8.$$

Each B_{ij} has rank 5 on an open set $\mathcal{V}_{ij} \subseteq \mathbb{R}^5$. The union of the corresponding six open sets $\mathcal{V}_{ij}$ contains the singular locus $\mathcal{V}_s$ as well as some regular points.

Consider now a point x_0 in $\mathbb{R}^5$. To define a homogeneous approximation procedure on the basis of the algorithm of Sect. II-B, we must instantiate Step 2 depending on the nature of x_0. If, to perform Step 2, we enforce the choice of B_r, we obtain a homogeneous approximation procedure AP_r defined on $\mathcal{V}_r$; if we enforce the choice of a B_{ij}, we obtain a homogeneous approximation procedure AP_{ij} defined on the corresponding $\mathcal{V}_{ij}$. In formulas:

$$\begin{aligned}
AP_r(x_0) &= (z_r, \Psi_r) \quad \text{for} \quad x_0 \in \mathcal{V}_r\\
AP_{ij}(x_0) &= \begin{cases} (z_{ij,r}, \Psi_{ij,r}) & \text{for} \quad x_0 \in \mathcal{V}_{ij} \cap \mathcal{V}_r \\ (z_{ij,s}, \Psi_{ij,s}) & \text{for} \quad x_0 \in \mathcal{V}_{ij} \cap \mathcal{V}_s. \end{cases}
\end{aligned}$$

At $x_0 \in \mathcal{V}_{ij} \cap \mathcal{V}_r$, both AP_r and AP_{ij} are defined and continuous. Instead, at $x_0 \in \mathcal{V}_{ij} \cap \mathcal{V}_s$, AP_{ij} is not continuous near x_0, while AP_r is not defined. Therefore, no homogeneous approximation procedure is continuous near a singular point — correspondingly, no such procedure gives a uniform distance estimation on the corresponding $\mathcal{V}_{ij}$.

In the following sections, we will show that nonhomogeneous NAs solve the above difficulties. Before, we propose an interesting example of system belonging to the family under consideration.

B. Example: The General 2-Trailer System

Consider the system shown in Figure 1, consisting of a mobile robot with unicycle kinematics towing two identical trailers of length ℓ, each hinged at a nonzero distance d from the previous wheel axle[4]. This vehicle is a particular instance ($n = 2$) of the so-called *general* n-trailer system.

[4]This kind of mechanical arrangement is referred to as *kingpin hitching* or *off-hooking* or *nonzero hooking*.

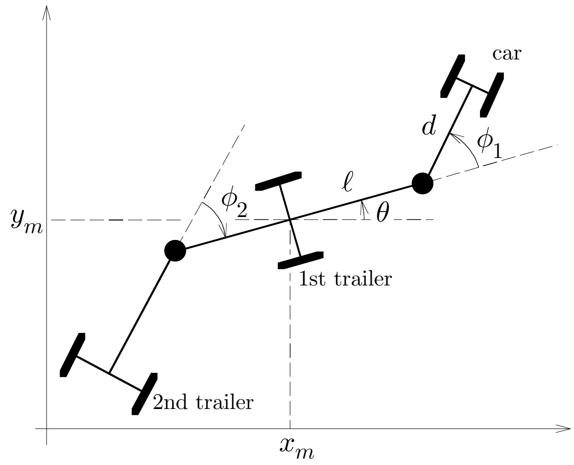

Fig. 1. The general two-trailer system

To display the symmetry of the system and make computations easier it is convenient to choose as reference point for the system the midpoint (x_m, y_m) of the first trailer wheel axle. This leads to the generalized coordinate vector $x = (x_m, y_m, \theta, \phi_1, \phi_2)$, where θ is the orientation of the first trailer with respect to the x axis, while ϕ_1 and ϕ_2 are the angles that the first and the second trailer form respectively with the car and the first trailer.

Under the convention that d defines the unit length and the assumption that $d = \ell$, the kinematic control system takes the form (6), where the input vector fields are given as

$$g_1(x) = \begin{pmatrix} \cos\theta & \sin\theta & 0 & \sin\phi_1 & -\sin\phi_2 \end{pmatrix}^T, \qquad g_2(x) = \begin{pmatrix} 0 & 0 & 1 & -1-\cos\phi_1 & 1+\cos\phi_2 \end{pmatrix}^T$$

and u_1, u_2 are related to the actual control inputs v_0 and ω_0 (respectively, the driving and steering velocity of the car) through a nonsingular input transformation [22].

If $\phi_1 = \pi$ or $\phi_2 = \pi$ the system is clearly not controllable. Define thus the state manifold as $\mathcal{M} = I\!\!R^2 \times S^1 \times (S^1 - \{\pi\})^2$. Simple computations show that, in this case, the value of B_r spans $T_{x_0}\mathcal{M}$ at points where $\phi_1 \neq \phi_2$, while the value of B_{46} spans $T_{x_0}\mathcal{M}$ at all points of $\mathcal{M}$. Hence, the system is controllable and its growth vector is $(2, 3, 5)$ at *regular* points ($\phi_1 \neq \phi_2$, $r = 3$) and $(2, 3, 4, 5)$ at *singular* points ($\phi_1 = \phi_2$, $r = 4$).

In conclusion, the general two-trailer system belongs exactly to the family of nonholonomic systems introduced in this section, for which nonhomogeneous nilpotent approximations offer a way to attack the control problem (e.g., see [22]).

V. Nonhomogeneous Approximation Procedure

With reference to system (6), we intend to show that, given a domain $\mathcal{V}_{ij}$, it is possible to devise a nonhomogeneous approximation procedure which is continuous at each point — whether regular or singular — of $\mathcal{V}_{ij}$. For illustration, we consider first the domain $\mathcal{V}_{46}$ (which is equal to the whole state space in the case of the general two-trailer system).

The key point is to modify the homogeneous approximation procedure given in Sect. II-B by *assigning* to the coordinate z_5 its maximum weight, i.e., $w_5 = 4$. The modified procedure, denoted by AP^{nh}_{46}, is detailed as follows (compare with Sect. II-B).

1. Set the weights to $1, 1, 2, 3, 4$.

2. Choose B_{46} as a set of vector fields.

3.–5. As in Sect. II-B.

6. Expand vector fields $g_i(z)$ in Taylor series and let

$$\bar{g}_i(z) = g_i^{(\leq -1)}(z),$$

having defined $g_i^{(\leq -1)}(z)$ as the sum of all the terms of weighted degree ≤ -1.

7. Define the approximate system Ψ_{46}^{nh} as

$$\dot{z}_j = \sum_{i=1}^{m} \bar{g}_{ij}(z_1, \ldots, z_{j-1}) u_i. \quad j = 1, \ldots, 5. \tag{7}$$

The inclusion of terms of weighted degree ≤ -1 in the definition of $\bar{g}_i(z)$ is due to the new assignment of weights. In particular, having now set $w_5 = 4$, ∂_{z_5} is of weighted degree -4. As a consequence, the weighted degree of a monomial $a_\alpha z_1^{\alpha_1} \ldots z_n^{\alpha_n} \partial_{z_j}$, computed with the new weights, is not equal to its order. Thus, at regular points the first monomials actually appearing in the Taylor expansion of the fifth component of $g_i(z)$ are of weighted degree $< w_5 - 1$. On the other hand, these monomials are automatically zero at singular points, for z_5 becomes there of order 4.

Theorem 1: AP_{46}^{nh} is an approximation procedure which depends continuously on x_0 in $\mathcal{V}_{46}$.

Proof. First, note that the system of coordinates provided by AP_{46}^{nh} is privileged. In fact, setting $w_5 = 4$ affects only the expression (computed through eq. (3)) of z_5 at regular points of $\mathcal{V}_{46}$, due to the appearance of additional higher-degree terms with respect to the same coordinate as provided there by AP_{46}. Since the addition of terms of higher degree does not affect the order of a coordinate, z_5 will still be of order 3 at regular points. At singular points, the coordinates provided by AP_{46}^{nh} and by AP_{46} are exactly the same. Hence, the coordinates z are privileged in $\mathcal{V}_{46}$ since they have order $(1, 1, 2, 3, 3)$ at regular points and $(1, 1, 2, 3, 4)$ at singular points.

We now show that Ψ_{46}^{nh} is a nilpotent approximation of system (6) in $\mathcal{V}_{46}$, expressed in the z coordinates. At singular points, Ψ_{46}^{nh} coincides with the homogeneous $\Psi_{46,s}$ obtained by applying AP_{46}. At regular points of $\mathcal{V}_{46}$, the order of privileged coordinates is $(1, 1, 2, 3, 3)$, and therefore the *homogeneous* approximation ofΨ_{46}^{nh} at $z = 0$, obtained by applying AP_{46} to eq. (7), coincides with $\Psi_{46,r}$. Hence, the homogeneous NA of Ψ_{46}^{nh} at $z = 0$ is also the homogeneous NA of system (6) at x_0, expressed in z. This proves condition *a)* of Definition 3. To prove condition *b)*, consider that the input vector fields $\bar{g}_1, \bar{g}_2$ of system (7) are, by construction, of weighted degree ≤ -1. As a consequence, their brackets of length ≥ 5 are of weighted degree ≤ -5. However, no monomial can be of weighted degree < -4, so that all brackets of length > 4 must be zero, i.e., system (7) is nilpotent of step 5.

Finally, the privileged coordinates z and the nilpotent approximation Ψ_{46}^{nh} are continuous in $\mathcal{V}_{46}$ by construction, and so is AP_{46}^{nh}. ∎

Ψ_{46}^{nh} has the same polynomial, triangular structure of the homogeneous NA (4). The distinctive feature of Ψ_{46}^{nh} is its nonhomogeneity: function $\bar{g}_{i5}(z_1, \ldots, z_4)$, $i = 1, 2$, is the sum of two polynomials of homogeneous degree 2 and 3, respectively. At singular points the coefficients of the monomials of homogeneous degree 2 vanish, so that only a polynomial of homogeneous degree 3 is left. We call AP_{46}^{nh} a *nonhomogeneous* approximation procedure and Ψ_{46}^{nh} a *nonhomogeneous* NA.

In a generic domain $\mathcal{V}_{ij}$, a nonhomogeneous approximation procedure AP_{ij}^{nh} is readily obtained by simply choosing B_{ij} as a set of vector fields in Step 2 above. The associated nilpotent approximation is denoted by Ψ_{ij}^{nh}.

The state space $I\!R^5$ of system (6) is given by the union of $\mathcal{V}_r$ and the six $\mathcal{V}_{ij}$'s defined in Sect. IV-A. In the particular case that one of the $\mathcal{V}_{ij}$'s covers the whole state space (i.e., if one of the B_{ij}'s provides a basis at every point), as happens for $\mathcal{V}_{46}$ in the general two-trailer system, then the nonhomogeneous approximation procedure AP_{ij}^{nh} provides at each point of the state space a system of privileged coordinates and a nilpotent approximation which depend continuously on the approximation point. In general, however, a basis B_{ij} valid on the whole space may not exist. Hence, there exists no approximation procedure (homogeneous or nonhomogeneous) that is defined and continuous everywhere. Still, around each point there exists at least one continuous approximation procedure: either AP_r or one of the AP_{ij}^{nh}.

For practical purposes — e.g., to compute an approximate steering control — one may also wish to associate a single, locally continuous approximation procedure to each point of the state space. To this end, it is possible to partition the state space into seven subsets with nonempty interior:

$$\mathcal{D}_r = \{x \in I\!R^5 :\ |\det \Gamma_r| \geq |\det \Gamma_{hl}|,\ \ h = 4,5,\ l = 6,7,8\}$$

$$\mathcal{D}_{ij} = \left\{ x \in I\!R^5, x \notin \mathcal{D}_r :\ \begin{array}{l} |\det \Gamma_{ij}| > |\det \Gamma_{hl}|,\ \text{for } \{hl\} < \{ij\} \\ |\det \Gamma_{ij}| \geq |\det \Gamma_{hl}|,\ \text{for } \{hl\} \geq \{ij\} \end{array} \right\},\quad i,h = 4,5,\ j,l = 6,7,8$$

where Γ_r and Γ_{hl} are the 5×5 matrices whose columns are respectively the vector of coordinates of the vector fields $\{g_1, \ldots, g_5\}$ and $\{g_1, g_2, g_3, g_h, g_l\}$ at x, and couples of indices have been ordered lexicographically. Each $\mathcal{D}_{ij}$ (respectively, $\mathcal{D}_r$) is included in $\mathcal{V}_{ij}$ (respectively, $\mathcal{V}_r$); therefore, by taking AP_{ij}^{nh} on $\mathcal{D}_{ij}$ and AP_r on $\mathcal{D}_r$ we have associated a unique continuous approximation procedure to each of the seven subsets.

VI. Uniform Estimation of the sub-Riemannian Distance

We now address the problem of obtaining a uniform estimate of the sub-Riemannian distance as a function of privileged coordinates. To this end, we first sketch the procedure for estimating uniformly the sub-Riemannian distance through the lifting method, and then show that an estimate based on privileged coordinates can be obtained by computing the relationship between the latter and the lifted privileged coordinates (such as eq. (11) below).

A. Lifting of the control system

We first desingularize the control system using the *lifting method*, based on the following result.

Lemma ([12]): Consider system (1) and $x_0 \in I\!R^n$. There exist an integer $\tilde{n} \geq n$; a neighborhood $\tilde{U} \subset I\!R^{\tilde{n}}$ of $(x_0, 0)$; coordinates (x, ξ) on $\tilde{U}$, where $\xi = (\xi_1, \ldots, \xi_{\tilde{n}-n})$; and smooth vector fields $\tilde{g}_i$ on $\tilde{U}$ in the form[5]

$$\tilde{g}_i(x, \xi) = g_i(x) + \sum_{j=1}^{\tilde{n}-n} b_{ij}(x, \xi) \partial_{\xi_j},$$

with the b_{ij}'s smooth functions on $I\!R^{\tilde{n}}$, such that the system defined by the *lifted* vector fields $\tilde{g}_1, \ldots, \tilde{g}_m$ satisfies the LARC and has no singular point in $\tilde{U}$.

Let $(x_1, 0)$ be a point in $\tilde{U}$ and $u(t)$, $t \in [0, T]$, be an input function. They define a trajectory in $I\!R^{\tilde{n}}$ steering the lifted system from $(x_1, 0)$ to (x_2, ξ), the solution at $t = T$ of the differential equation:

$$(\dot{x}(t), \dot{\xi}(t)) = \sum_{i=1}^{m} \tilde{g}_i(x(t), \xi(t))\, u_i(t),$$

[5]With a little abuse of notation, we denote by g_i the vector fields obtained by extending the input vector fields of system (1) with $\tilde{n} - n$ coordinates equal to zero.

with initial condition $(x(0), \xi(0)) = (x_1, 0)$. Using the definition of the lifted vector fields, we write these equations as

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{m} g_i(x(t))\, u_i(t) \\ \dot{\xi}_j(t) &= \sum_{i=1}^{m} b_{ij}(x(t), \xi(t))\, u_i(t), \quad j = 1, \dots, \tilde{n} - n, \end{aligned}$$

with $x(0) = x_1$, $\xi_j(0) = 0$. The first equation represents the original system in $I\!R^n$. Therefore, the canonical projection of the trajectory in $I\!R^{\tilde{n}}$ associated to $u(t)$ and steering the lifted system from $(x_1, 0)$ to (x_2, ξ) is the trajectory in $I\!R^n$ associated to the same control $u(t)$ and steering the original system from x_1 to x_2. In particular, the two trajectories have the same length.

In view of definition (2), the sub-Riemannian distance between x_1 and x_2 in a neighborhood of x_0 is

$$d(x_1, x_2) = \inf_{\xi \in I\!R^{\tilde{n}-n}} \tilde{d}((x_1, 0), (x_2, \xi)), \tag{8}$$

where $\tilde{d}$ denotes the sub-Riemannian distance for the lifted system.

B. Distance estimation

As in Sect. V, we consider for illustration the case $x_0 \in \mathcal{V}_{46}$. Our objective is to build a regular system in some space $I\!R^{\tilde{n}}$ such that its canonical projection on $I\!R^5$ near x_0 coincides with the considered system. To this end, let $b_1(x, \xi)$ and $b_2(x, \xi)$ be C^∞ functions on $I\!R^5 \times I\!R$ and set

$$\tilde{g}_i(x, \xi) = g_i(x) + b_i(x, \xi)\partial_\xi, \qquad i = 1, 2. \tag{9}$$

For a generic (for the C^3 topology) choice of the functions b_i, the lifted system defined by $\tilde{g}_1, \tilde{g}_2$ on $I\!R^{\tilde{n}} = I\!R^6$ will have growth vector (2,3,5,6) at $(x_0, 0)$. Hence, this system satisfies the LARC and has no singular point in a neighborhood $\tilde{U}_{x_0}$ of $(x_0, 0)$.

Consider the first 8 elements of a P. Hall basis as given in Sect. IV-A and the evaluation map $E_{\tilde{g}}$ which assigns to each element of the Lie Algebra in the indeterminates $\{X_1, X_2\}$ the vector field obtained by plugging in the $\tilde{g}_i$, $i = 1, 2$, for the corresponding X_i. Denoting by $\tilde{g}_3, \dots, \tilde{g}_8$ the vector fields given by $\tilde{g}_j = E_{\tilde{g}}(X_j)$, $j = 3, \dots, 8$, consistently with eq. (9) we have

$$\tilde{g}_i(x, \xi) = g_i(x) + b_i(x, \xi)\partial_\xi, \qquad i = 3, \dots, 6.$$

Reducing (if needed) $\tilde{U}_{x_0}$ so that $\tilde{U}_{x_0} \subset \mathcal{V}_{46} \times I\!R$, and using the genericity of b_1 and b_2, we can assume that $\{\tilde{g}_1(x, \xi), \dots, \tilde{g}_6(x, \xi)\}$ has rank 6 at any point $(x, \xi) \in \tilde{U}_{x_0}$.

Let $(x_1, 0) \in \tilde{U}_{x_0}$. We want to compute privileged coordinates in $I\!R^6$ around $(x_1, 0)$ for the lifted control system, and compare them with $z_1, \dots, z_5, \xi$, where the z_i's are the coordinates constructed in Section V. To this end, we follow steps 1–4 of the procedure given in Sect. V.

1. Set the weights to $1, 1, 2, 3, 4, 4$.
2. For the choice of the vector fields, note first that, being $x_1 \in \mathcal{V}_{46}$, we have

$$g_5(x_1) = \rho_1 g_1(x_1) + \ldots + \rho_4 g_4(x_1) + \rho g_6(x_1), \tag{10}$$

where $\rho = 0$ if the point x_1 is singular. Set then $\tilde{g}_5' = \tilde{g}_5 - \rho_1 \tilde{g}_1 - \ldots - \rho_4 \tilde{g}_4$ and choose the vector fields $\tilde{g}_1, \dots, \tilde{g}_5', \tilde{g}_6$. At $(x_1, 0)$, the following relation holds

$$\begin{aligned} \tilde{g}_i(x_1, 0) &= g_i(x_1, 0) + b_i^0 e_\xi, \qquad i = 1, \dots, 4, 6, \\ \tilde{g}_5'(x_1, 0) &= \rho g_6(x_1, 0) + b_5^0 \partial_\xi, \end{aligned}$$

where $b_i^0 = b_i(x_1, 0)$ for $i = 1, \ldots, 6$. Note that $\beta = b_5^0 - \rho b_6^0$ is nonzero.

3. Compute local coordinates $\tilde{y} = (\tilde{y}_1, \ldots, \tilde{y}_6)$ as

$$\begin{pmatrix} x - x_1 \\ \xi \end{pmatrix} = \tilde{\Gamma}\tilde{y},$$

where $\tilde{\Gamma}$ is the 6×6 matrix whose columns are the vector of coordinates of the vector fields $\tilde{g}_1, \ldots, \tilde{g}_6$ at $(x_1, 0)$. Noticing that

$$x - x_1 = \Gamma_{46} \;\; y,$$

where $y = (y_1, \ldots, y_5)$ and Γ_{46} is the 5×5 matrix whose columns are the vectors of coordinates of the vector fields $g_1, \ldots, g_4, g_6$ at x_1, we obtain

$$\begin{aligned} \tilde{y}_i &= y_j, \qquad j = 1, \ldots, 4, \\ \tilde{y}_5 &= \frac{1}{\beta}(\xi - b_6^0 y_5 - b_4^0 y_4 - \ldots - b_1^0 y_1), \\ \tilde{y}_6 &= y_5 - \frac{\rho}{\beta}(\xi - b_6^0 y_5 - b_4^0 y_4 - \ldots - b_1^0 y_1) = y_5 - \rho \tilde{y}_5. \end{aligned}$$

4. Define privileged coordinates $(\tilde{z}_1, \ldots, \tilde{z}_6)$ around $(x_1, 0)$ using eq. (3).

Since $\tilde{g}_i y_j = g_i y_j$ for $i \leq 4$, we have $\tilde{z}_j = z_j$ for $j \leq 4$. The last two coordinates have the form:

$$\begin{aligned} \tilde{z}_5 &= \frac{\xi}{\beta} + \psi_5(z_1, \ldots, z_5), \\ \tilde{z}_6 &= z_5 - \rho \tilde{z}_5. \end{aligned} \tag{11}$$

The use of the privileged coordinates provided by the nonhomogeneous approximation procedure AP_{46}^{nh} has led to an expression of coordinate $\tilde{z}_6$ which depends only on z_5 and $\tilde{z}_5$. These coordinates are used in the proof of the following theorem to estimate the sub-Riemannian distance $\tilde{d}$ on $\tilde{U}_{x_0}$.

Theorem 2: Let $S \subset \mathcal{V}_{46}$ be a compact set. There exist c, C and $\epsilon > 0$ such that, for all $x_1 \in S$ and all x with $d(x_1, x) < \epsilon$,

$$cf'(z) \leq d(x_1, x) \leq Cf'(z), \tag{12}$$

where

$$f'(z) = |z_1| + |z_2| + |z_3|^{1/2} + |z_4|^{1/3} + \min\left(\left|\frac{z_5}{\rho}\right|^{1/3}, |z_5|^{1/4}\right), \qquad \rho = \frac{\det(\Gamma_r)}{\det(\Gamma_{46})}. \tag{13}$$

and Γ_r is the 5×5 matrix whose columns are the vector of coordinates of $g_1, \ldots, g_5$ at x_1.

Proof. Consider $x_0 \in \mathcal{V}_{46}$. We will first prove the result for a compact neighborhood N_{x_0} of x_0 such that $N_{x_0} \times \{0\} \subset \tilde{U}_{x_0}$. At any point $\tilde{x}_1 = (x_1, 0) \in N_{x_0} \times \{0\}$, the Ball-Box Theorem guarantees the existence of constants $\tilde{c}_1$, $\tilde{C}_1$ and $\tilde{\epsilon}_1 > 0$ such that an inequality like (5) holds if $\tilde{d}(\tilde{x}_1, \tilde{x}) < \tilde{\epsilon}_1$. Moreover, $(x_1, 0)$ is a regular point and, by construction, the system of privileged coordinates $\tilde{z}_1, \ldots, \tilde{z}_6$ around $(x_1, 0)$ varies continuously with x_1. Then, $\tilde{c}_1$, $\tilde{C}_1$ and $\tilde{\epsilon}_1$ are continuous functions of x_1 and they have finite, nonzero extrema on the compact set N_{x_0}. As a consequence, there exist $\tilde{c}$, $\tilde{C}$ and $\tilde{\epsilon} > 0$ such that, for any $x_1 \in N_{x_0}$ and any $\tilde{x} = (x, \xi)$ such that $\tilde{d}(\tilde{x}_1, \tilde{x}) < \tilde{\epsilon}$, it is

$$\tilde{c} f(\tilde{z}) \leq \tilde{d}(\tilde{x}_1, \tilde{x}) \leq \tilde{C} f(\tilde{z}), \tag{14}$$

where

$$f(\tilde{z}) = |\tilde{z}_1| + |\tilde{z}_2| + |\tilde{z}_3|^{1/2} + |\tilde{z}_4|^{1/3} + |\tilde{z}_5|^{1/3} + |\tilde{z}_6|^{1/4}.$$

According to eq. (8), it is

$$d(x_1, x) = \inf_{\xi \in I\!R} \tilde{d}((x_1, 0), (x, \xi)).$$

Being $\partial\tilde{z}_5/\partial\xi = 1/\beta$ nonzero and using eq. (11), we may write

$$\inf_{\xi \in I\!R} f(\tilde{z}) = \inf_{\tilde{z}_5 \in I\!R} \left(|z_1| + \ldots + |z_4|^{1/3} + |\tilde{z}_5|^{1/3} + |z_5 - \rho\tilde{z}_5|^{1/4} \right).$$

The infimum is attained at $\tilde{z}_5 = z_5/\rho$ if $|z_5| \le \rho^4$ and at $\tilde{z}_5 = 0$ if $|z_5| \ge \rho^4$. This, together with the estimate (14) of $\tilde{d}$, gives the estimate of $d(x_1, x)$, with $c = \tilde{c}$ and $C = \tilde{C}$. The expression of ρ is easily derived from (10).

So far, the result has been established on a compact neighborhood N_{x_0} of each point x_0 in $\mathcal{V}_{46}$. Now, let S be a compact subset of $\mathcal{V}_{46}$. The union of the interiors V_{x_0} of N_{x_0}, $x_0 \in S$, is a covering of S by open sets. We can then extract a finite covering $\cup V_i$ of S. Equation (12) holds on each V_i with constants c_i, C_i and ϵ_i. Setting $\epsilon = \min_i \epsilon_i$, $c = \min_i c_i > 0$, and $C = \max_i C_i$, we obtain the required result. ■

Note the following points.

- The estimate does not depend on the choice of the lifted system.
- When x_1 is a singular point, the continuous function ρ equals zero and Theorem 2 is simply the Ball-Box Theorem at a singular point. On the other hand, when x_1 is regular and far enough from the singular locus, it may be certainly assumed that $\rho > \epsilon$ (reducing ϵ if needed). In this case, condition $d(x_1, x) < \epsilon$ implies $|z_5| \le \rho^4$, and Theorem 2 turns out to be the Ball-Box Theorem at a regular point.
- A uniform estimate of the form (12–13) holds for compact subsets of the generic $\mathcal{V}_{ij}$, with the privileged coordinates defined by AP^{nh}_{ij} and $\rho_{ij} = \det\Gamma_r/\det\Gamma_{ij}$ in place of ρ. The same is true on compact subsets of $\mathcal{V}_r$, with the privileged coordinates defined by AP_r and $\rho_r = 1$ in place of ρ, because by doing so the estimate (12–13) coincides with that of the classical Ball-Box theorem.

Analogously to the discussion at the end of Sect. V, we note that if $\mathcal{V}_{46}$ covers the whole state space the above theorem directly provides a uniform estimation of the sub-Riemannian distance on $I\!R^5$. Even in the general case, however, it is possible to obtain the same result as follows.

Given any compact subset $K \subset I\!R^5$, we can write $K = \left(\bigcup_{i,j} K_{ij}\right) \cup K_r$, where $K_{ij} = K \cap \mathcal{D}_{ij}$ and $K_r = K \cap \mathcal{D}_r$. As noted above, an estimate in the form (12–13) holds on each K_{ij} as well as K_r, and therefore a uniform distance estimation over K is readily obtained by computing the appropriate extremal values of c, C and ϵ over the subset.

VII. Conclusions

A drawback of homogeneous nilpotent approximations of nonlinear control systems is their discontinuous dependence on the approximation point in the neighborhood of singularities. With reference to the family of 5-dimensional, 2-input driftless systems having growth vector $(2, 3, 5)$ at regular points and $(2, 3, 4, 5)$ at singular points, it has been shown that such discontinuous behavior is avoided by a different kind of approximation. In particular, it has been shown that the state space may be partitioned in a finite collection of open domains, such that in each domain it is possible to define a procedure yielding privileged coordinates that vary continuously with the approximation point. Expanding in Taylor series the dynamics written in these coordinates, and retaining certain higher-order terms (with respect to the homogeneous approximation procedure), the procedure also provides a nonhomogeneous nilpotent approximation which shares the same continuity property over the domain. Moreover, a generalization of the Ball-Box Theorem can be given so as to derive from the new privileged coordinates an estimate of the sub-Riemannian distance which is uniform over any compact subset of the state space.

A natural question arises concerning the possibility of extending the proposed technique beyond the considered family of nonlinear systems. Preliminary results [13] indicate that an extension of the nonhomogeneous nilpotent approximation procedure — and of the corresponding uniform distance estimation — to the general driftless dynamics (1) is possible under generic hypothesis.

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